Q1. (**True or False**) Please circle the correct answer. Each question worths 0.5 points. (You do not have to explain your answer.)

In all the statements below, V is a finite dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $T : V \to V$ is a linear operator on V.

(i) If $A \in M_{2\times 2}(\mathbb{C})$ is symmetric, i.e. $A^t = A$, then A must be normal.

TRUE FALSE

(ii) Every unitary operator is normal.

TRUE FALSE

(iii) An orthogonal projection is uniquely determined by its range.

TRUE FALSE

(iv) If T is unitary, then T^* is also unitary.

TRUE FALSE

(v) If T is an orthogonal operator, then all the eigenvalues of T are equal to 1.

TRUE FALSE

(vi) In $\mathbb{R}^2$, the composition of a rotation with a reflection is a rotation.

TRUE FALSE

(vii) The matrix $A = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \in M_{2\times 2}(\mathbb{C})$ has an orthonormal eigenbasis for $\mathbb{C}^2$.

TRUE FALSE

(viii) Let $\mathbb{F} = \mathbb{R}$. If β and γ are orthonormal bases for V, then the change of coordinate matrix Q from β to γ is an orthogonal matrix.

TRUE FALSE

Q2. **(Short Questions)** Each question worth 1 point. (You do not have to explain your answer.)

In all the statements below, V is a finite dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $T : V \to V$ is a linear operator on V.

(i) If $\dim V = 5$ and $\text{nullity}(T) = 2$, then what is $\text{nullity}(T^*)$? *(Hint: recall that $N(T)^\perp = R(T^*)$.)*

Answer: 2

(ii) Suppose that a matrix $A \in M_{2\times 2}(\mathbb{C})$ is normal and has distinct eigenvalues 1 and $1+i$. Find all the eigenvalues of A^*.

Answer: 1, 1−i

(iii) Suppose that $[T]_\beta = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ for an orthonormal basis β of V. Write down the matrix $[T^*]_\beta$.

Answer: $\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$

(iv) If $A \in M_{n\times n}(\mathbb{R})$ is an orthogonal matrix, find $\det(A^2)$.

Answer: 1

(v) Write down a vector in $\mathbb{R}^2$ which is orthogonal to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with respect to the inner product defined by

$$\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rangle = x_1y_1 + x_2y_1 + x_1y_2 + 2x_2y_2.$$

Answer: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(vi) Suppose $A \in M_{2\times 2}(\mathbb{R})$ is symmetric with eigenvalues 1 and -1. Find $\text{tr}(A^tA)$.

Answer: 2

Q.3 (10 points) Let $A \in M_{3\times 3}(\mathbb{R})$ be the matrix

$$A = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

(a) (5 points) It is known that -1 is an eigenvalue of A. Find an orthonormal basis for the eigenspace E_{-1} of A.

$$A-(-1)I = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$E_{-1} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 - x_2 + x_3 = 0 \right\}.$$

$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for E_{-1}.

Next, we apply the Gram-Schmidt process.

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \end{pmatrix}.$$

Normalizing v_1, v_2, we obtain an orthonormal basis for E_{-1}: $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}.$

(b) (5 points) Find an orthogonal matrix $Q \in M_{3\times 3}(\mathbb{R})$ such that $Q^t AQ$ is diagonal.

$\det(A - \lambda I) = -\lambda^3 + 3\lambda + 2 = -(\lambda+1)^2(\lambda-2)$.

The eigenvalues of A are -1, 2.

$$A - 2I = \begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$E_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : \quad x_1 + x_2 = 0, \quad x_2 + x_3 = 0 \right\}.$$

$\left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$ is an orthonormal basis for E_2.

Using part (a), $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$ is an orthonormal eigenbasis of A for $\mathbb{R}^3$.

Let $Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$. Then Q is orthogonal.

and $Q^T A Q = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 2 \end{pmatrix}$ is diagonal.

Q.4 Let V be a finite dimensional complex inner product space.

(a) (2 points) Prove that for all $x, y \in V$,

$$\|x+y\|^2 = \|x\|^2 + 2\Re\langle x, y\rangle + \|y\|^2,$$

where $\Re(z)$ denotes the real part of the complex number z.

For all $x, y \in V$,

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$
$$= \langle x, x+y\rangle + \langle y, x+y\rangle$$
$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$
$$= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2$$
$$= \|x\|^2 + 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2.$$

(b) (4 points) If $T: V \to V$ is a self adjoint operator, prove that for all $x \in V$,

$$\|T(x) + ix\|^2 = \|T(x)\|^2 + \|x\|^2.$$

(*Hint: consider the complex number $\langle Tx, ix\rangle$ and use (a).*) Use this to deduce that $T + iI$ is invertible.

For all $x \in V$, $\langle Tx, ix\rangle = -i\langle x, T^* x\rangle$

$$= -i\langle x, Tx\rangle$$
$$= -\langle ix, Tx\rangle$$
$$= -\overline{\langle Tx, ix\rangle}.$$

Hence $\mathrm{Re}\langle Tx, ix\rangle = 0$ and by part (a), we have

$\|Tx + ix\|^2 = \|Tx\|^2 + \|ix\|^2 = \|Tx\|^2 + \|x\|^2, \ \forall x \in V.$

Now, if $(T+iI)x = 0$, then $\|Tx\|^2 + \|x\|^2 = 0$.

Since $\|Tx\|^2 \geq 0$, $\|x\|^2 \geq 0$, this implies that $\|x\| = 0$,

and $x = 0$. Hence we have shown that $(T+iI)$ is injective.

Since V is finite dimensional, $T+iI$ must be invertible.

Q.5 (a) (2 point) Write down a matrix $A \in M_{2\times 2}(\mathbb{R})$ where there does not exists $B \in M_{2\times 2}(\mathbb{R})$ such that $B^2 = A$. Explain your answer.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that $\det A = -1$.

If $\exists\, B \in M_{2\times 2}(\mathbb{R})$ such that $B^2 = A$, then

$\det A = (\det B)^2 \geq 0$, which is a contradiction.

(b) (2 points) Let $A \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix whose eigenvalues are all real and nonnegative. Prove that there exists a symmetric matrix $B \in M_{n\times n}(\mathbb{R})$ such that $B^2 = A$. *Hint: Use spectral theorem.*

Since A is symmetric, there exists an orthogonal matrix Q s.t. $Q^T A Q = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$.

By assumption, we have $\lambda_i \geq 0$, $i = 1, 2, \dots, n$.

Now, let $C = \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix}$ and $B = Q C Q^T$.

Then $B^T = Q^{TT} C^T Q^T = Q C Q^T = B$

and $B^2 = Q C Q^T Q C Q^T = Q C^2 Q^T = A$.

—END OF MIDTERM 2—